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METHOD OF ORTHOGONAL EXPANSIONS ON THE DOMAIN BOUNDARY IN THREE-DIMENSIONAL PROBLEMS OF THE LINEAR THEORY OF ELASTICITY

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There is proposed a new method of solving three-dimensional boundary value problems of the linear theory of elasticity, related to the Castigliano principle. Construction of the solution reduces to determination of the projection of an element satisfying the equation of elasticity theory in the domain, and the boundary condition on a free part of the boundary, in the subspace of boundary values of solutions of the boundary value problem of the theory of elasticity with inhomogeneous natural boundary conditions.

1. Formulation of the problem. Scheme of the method. The Castigliano principle, to which the method of orthogonal projections reduces in problems of the theory of elasticity [1], is that out of all the stress tensors satisfying the equilibrium equations in the domain and a boundary condition on a free part of the boundary, the elastic stress tensor expressed by the inequality

$$\|\mathbf{R}^*\|_\Sigma^2 = \|\mathbf{R}_0\|_\Sigma^2 + \|\mathbf{R}^* - \mathbf{R}_0\|_\Sigma^2 \geqslant \|\mathbf{R}_0\|_\Sigma^2 \tag{1.1}$$

communicates the least strain potential energy to the body.

Here $\mathbf{R}^*$ is an arbitrary stress tensor satisfying the equilibrium equations of an elastic medium, and the boundary condition on a free part of the boundary, $\mathbf{R}_0$ is the elastic stress tensor, Σ is the Hilbert space generating a set of tensors with finite energy integral. This space is decomposable [1] into an orthogonal sum $\Sigma = \Sigma_1 \oplus \Sigma_2$, where Σ_1 is a subspace of tensors $\mathbf{R}'$ associated with the displacement vector $\mathbf{u}'$ satisfying the boundary condition on the fixed part of the boundary, and Σ_2 is a subspace of tensors $\mathbf{R}''$ related to the displacement vector $\mathbf{u}''$ satisfying the homogeneous equilibrium equations and the boundary condition on the free part of the boundary. Then the condition of orthogonality of the tensors $(\mathbf{R}', \mathbf{R}'')_\Sigma = 0$ and $\mathbf{R}^* = \mathbf{R}' + \mathbf{R}''$, $\mathbf{R}' \in \Sigma_1$, $\mathbf{R}'' \in \Sigma_2$ holds.

We represent [2] an arbitrary displacement vector $\mathbf{u}^*(x)$ related to the tensor R^* satisfying the elasticity theory equation $\mathbf{A}\mathbf{u}^* = \mathbf{K}$ and the boundary condition on the free part of the boundary S, in the form of the sum $\mathbf{u}^*(x) = \mathbf{u}_0(x) + \boldsymbol{\varphi}_0(x)$, where $\mathbf{u}_0(x)$ is the energy solution of the fundamental boundary value problem of elasticity theory, and the vector $\boldsymbol{\varphi}_0(x)$ is the solution of the additional boundary value problem

$$\mathbf{A}\boldsymbol{\varphi}_0 = 0 \text{ in } G, \quad \mathbf{t}^{(\nu)}(\boldsymbol{\varphi}_0)|_S = \mathbf{t}^{(\nu)}(\mathbf{u}^*)|_S \tag{1.2}$$

$$\mathbf{A}\mathbf{u} \equiv -\sum_{i,k,l,m=1}^{3} \frac{\partial}{\partial x_i}\left[c_{iklm}(x)\,\varepsilon_{lm}(\mathbf{u})\right]\mathbf{x}_k^{(0)}, \quad x \in G$$

$$\varepsilon_{lm}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l}\right), \quad l, m = 1, 2, 3$$

$$\mathbf{t}^{(\nu)}(\mathbf{u}) = \sum_{i,k,l,m=1}^{3} c_{iklm}(x)\,\varepsilon_{lm}(\mathbf{u})\cos(\nu, x_i)\,\mathbf{x}_k^{(0)}$$

Here $\mathbf{A}$ is the differential operator of anisotropic elasticity theory, $c_{iklm}(x)$ are the anisotropy coefficients of the medium which satisfy symmetry conditions [1], $\varepsilon_{lm}(\mathbf{u})$ are strain tensor components, $x_k^{(0)}$ are directions of the axes x_k; $G \subset E_3$ is a bounded domain occupied by the elastic medium, with a two-dimensional sufficiently smooth surface S as boundary, and $\mathbf{t}^{(\nu)}(\mathbf{u})$ is the vector of the stresses acting on the area of the surface S.

The solution of the second inhomogeneous problem (1.2) of elasticity theory will be understood in the generalized sense as a vector $\varphi_0 \in W_2^1(G)$ satisfying the integral identity

$$2\int_G W(\varphi_0, \mathbf{u})\,dG - J_S(\mathbf{u}, \mathbf{u}^*) = 0, \quad \forall \mathbf{u} \in W_2^1(G) \tag{1.3}$$

$$W(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\sum_{i,k,l,m=1}^{3} c_{iklm}\varepsilon_{lm}(\mathbf{u})\,\varepsilon_{ik}(\mathbf{v}), \quad J_S(\mathbf{u}, \mathbf{v}) = \int_S \mathbf{u}\cdot\mathbf{t}^{(\nu)}(\mathbf{v})\,ds$$

Here $W(\mathbf{u}, \mathbf{u})$ is a positive definite quadratic form in the components of the elastic strain tensor. The Betti formula [1] based on the fact that [3] for $\mathbf{u} \in W_2^1(G)$ and $\mathbf{A}\mathbf{u} \in L_2(G)$ the trace $\partial\mathbf{u}/\partial\nu \in W_2^{-1/2}(S)$ is defined uniquely ($\partial/\partial\nu$ is the derivative with respect to the external normal ν to S) is used in (1.3). Upon compliance with the conditions

$$\int_S \mathbf{t}^{(\nu)}(\mathbf{u}^*)\,ds = \int_S \mathbf{t}^{(\nu)}(\mathbf{u}^*)\cdot\mathbf{r}\,ds = 0 \tag{1.4}$$

$$\int_G \varphi_0\,dG = \int_G \operatorname{rot}\varphi_0\,dG = 0 \tag{1.5}$$

zero is not an eigennumber of the problem (1.2) [2], and the operator $(\mathbf{A}, \mathbf{t}^{(\nu)})$ is an isomorphism from $W_2^1(G)$ on $L_2(G) \times W_2^{-1/2}(S)$, i.e., the problem (1.2) has a unique solution in the sense (1.3) for any vector $\mathbf{t}^{(\nu)}(\mathbf{u}^*) \in W_2^{-1/2}(S)$ satisfying (1.4).

The construction of the method of orthogonal expansions on the domain boundary reduces to the construction of a subspace of traces $W(S) \subset W_2^{1/2}(S)$ in which the traces $\mathbf{u}_0|_S$ and $\varphi_0|_S$, which make the surface integral J_S in (1.3) vanish, are orthogonal for all the boundary conditions of elasticity theory problems ($W_2^{1/2}(S)$ is the Sobolev–Slobodetskii space, and $W_2^{-1/2}(S)$ is its dual [3, 4]).

The problem of constructing such a space reduces to the problem of constructing the equipment of the fundamental space $W_0 = L_2(S) = (L_2(S))'$ (the prime denotes duality) of a dual pair of spaces of the traces $W \subset W_0 \subset W'$ for which the operator $\mathbf{T}$ generated by the boundary form J_S as the duality ratio on $W_2^{1/2}(S) \times W_2^{-1/2}(S)$ (which is an extension of the scalar product in $L_2(S)$ because of the continuous and compact imbeddings $W_2^{1/2}(S) \subset L_2(S) \subset W_2^{-1/2}(S)$) is an isometry from W into W'. This latter is determined by the relation from the known Riesz theorem [5]

$$(\mathbf{u}, \mathbf{v})_W = (\mathbf{u}, \mathbf{T}\mathbf{v})_{W_0} = (\mathbf{T}\mathbf{u}, \mathbf{T}\mathbf{v})_{W'}, \quad \forall \mathbf{u}, \mathbf{v} \in W \tag{1.6}$$

Then hypotheses about the method of orthogonal expansions in the domain boundary result from the properties of the operator $\mathbf{T}$. If $\mathbf{T} = \mathbf{T}_0$, where $\mathbf{T}_0$ is the canonical isometry from $W_2^{1/2}(S)$ into $W_2^{-1/2}(S)$ (the operator is called partially isometric if it is isometric on the orthogonal complement to its core [5]), then the operator $\mathbf{T}^*\mathbf{T}$ is the projector on the subspace $(\ker \mathbf{T})^\perp$, the initial space of partial isometry [5], and $\mathbf{T}$ is the unitary mapping of $(\ker \mathbf{T})^\perp$ on $W_2^{-1/2}(S)$ ($\ker \mathbf{T}$: $\mathbf{u} \in W_2^{1/2}(S) \mid \mathbf{T}\mathbf{u} = 0$). In this case the orthogonal decomposition of the space $W_2^{1/2}(S)$ holds

$$W_2^{1/2}(S) = \ker \mathbf{T} \oplus (\ker \mathbf{T})^\perp, \quad \dim\ker \mathbf{T} < \infty \tag{1.7}$$

The subspace $(\ker \mathbf{T})^\perp$ is a subspace $W(S)$ of boundary values of the vectors φ_0, the solutions of the additional problem (1.2). Then, the method of solving the fundamental boundary value problem of elasticity theory for $\mathbf{u}_0(x)$, which is the fundamental method of orthogonal expansions on the domain boundary, follows from the orthogonal expansion of the space $W_2^{1/2}(S)$. Let $\mathbf{u}^*(x)$ be an arbitrary vector satisfying the elasticity theory equation $\mathbf{A}\mathbf{u}^* = \mathbf{K}$ in the domain G ($\mathbf{K}(x)$ is the volume force vector) and the boundary condition on the free suface S such that $\mathbf{u}^*|_S \in W_2^{1/2}(S)$. Projecting the vector $\mathbf{u}^*$ onto the subspace $(\ker \mathbf{T})^\perp = W$ and subtracting the projection $\mathbf{T}^*\mathbf{T}\mathbf{u}^* = \varphi_0$ from the vector $\mathbf{u}^*$, we obtain $\mathbf{u}_0 = \mathbf{u}^* - \varphi_0$. Therefore, the element $\mathbf{u}_0$ satisfies the equation $\mathbf{A}\mathbf{u}_0 = \mathbf{K}$ in the domain G, and the condition $\mathbf{u}_0|_S = (\mathbf{u}^* - \varphi_0)|_S \in \ker \mathbf{T}$ holds on the boundary S. The imbedding is here understood in the sense $(\mathbf{T}\mathbf{u}_0, \mathbf{v})_{0,S} = 0$, $\forall \mathbf{v} \in W$, therefore, $\mathbf{T}\mathbf{u}_0 = 0$ and since $\mathbf{T}$ is the unitary mapping of $W(S)$ onto $W_2^{-1/2}(S)$, then $\mathbf{u}_0|_S = 0$.

The problem of constructing the method of orthogonal expansions on the domain boundary therefore reduces to proving the following auxiliary propositions:

a) The operator $\mathbf{T}$ is a partial isometry from $W_2^{1/2}(S)$ into $W_2^{-1/2}(S)$;

b) The initial space of this partial isometry is a subspace of the boundary values of the solutions of the additional problem (1.2).

Let us note that the constructions expounded have points of contiguity with the abstract scheme of the method of orthogonal projections of the solution for boundary value problems of second order elliptic equations [1]. In particular, the condition of applicability of the method of orthogonal projections is common, i.e., we represent the positive operator in the form of the product of two conjugate operators. As will be shown below, $\mathbf{T}^*\mathbf{T}$ is such a product. There are also certain analogies with [6].

2. **Construction of the Space $W(S) = W_2^{*1/2}(S)$.** Let us prove the propositions a) and b) formulated in Sect. 1.

Let the vector functions $\mathbf{u}$ and $\mathbf{v}$ satisfy the equation $\mathbf{A}\mathbf{v} = 0$ in the domain

G. Then, according to the Betti formula [1], we obtain

$$2\int_G W(\mathbf{u},\mathbf{v})\,dG = J_S(\mathbf{u},\mathbf{v}) \tag{2.1}$$

It hence follows that the bilinear boundary form $J_S(\mathbf{u},\mathbf{v})$ is symmetric, and if the vector functions $\mathbf{u}$ and $\mathbf{v}$ satisfy the condition (1.5), then the corresponding quadratic form $J_S(\mathbf{u},\mathbf{u})$ will be positive. According to the Riesz theorem on the general form of a linear continuous functional in Hilbert space [7] for the bilinear form $\langle\mathbf{u},\mathbf{t}^{(\nu)}(\mathbf{v})\rangle = J_S(\mathbf{u},\mathbf{v})$ (the angular brackets signify the duality ratio on $W_2^{1/2}(S)\times W_2^{-1/2}(S)$), which is separately continuous because of the generalized Schwartz inequality $|\langle\mathbf{u},\mathbf{t}^{(\nu)}(\mathbf{v})\rangle| \leqslant \|\mathbf{u}\|_{1/2,S}\,\|\mathbf{t}^{(\nu)}(\mathbf{v})\|_{-1/2,S}$, the following representation holds:

$$J_S(\mathbf{u},\mathbf{v}) = (\mathbf{u},\mathbf{T}\mathbf{v})_{0,S}, \quad \forall\mathbf{u}\in W_2^{1/2}(S) \tag{2.2}$$

Here $\mathbf{T}$ is some linear continuous operator determined in the whole space $W_2^{1/2}(S)$ and acting in $W_2^{-1/2}(S)$ such that

$$\|\mathbf{T}\mathbf{v}\|_{-1/2,S} = \|\mathbf{t}^{(\nu)}(\mathbf{v})\|_{-1/2,S} \leqslant c\,\|\mathbf{v}\|_{1/2,S}, \quad c>0$$

The compactness of the operator $\mathbf{T}$ follows from the compactness of the imbedding of $W_2^{1/2}(S)$ into $W_2^{-1/2}(S)$ and there results from the symmetry of the bilinear form $J_S(\mathbf{u},\mathbf{v})$ and the positivity of the quadratic form $J_S(\mathbf{u},\mathbf{u})$ that the operator $\mathbf{T}$ is symmetric and positive. We define the Hilbert conjugate operator $\mathbf{T}^*$ acting from $W_2^{-1/2}(S)$ into $W_2^{1/2}(S)$ for the bounded operator $\mathbf{T}$ by the equality

$$(\boldsymbol{\psi},\mathbf{T}\boldsymbol{\varphi})_{-1/2,S} = (\mathbf{T}^*\boldsymbol{\psi},\boldsymbol{\varphi})_{1/2,S}, \quad \forall\boldsymbol{\psi}\in W_2^{-1/2}(S), \quad \boldsymbol{\varphi}\in W_2^{1/2}(S)$$

Setting $\boldsymbol{\psi} = \mathbf{T}\mathbf{v}\in W_2^{-1/2}(S)$, $\boldsymbol{\varphi} = \mathbf{u}\in W_2^{1/2}(S)$, we obtain

$$(\mathbf{T}^*\mathbf{T}\mathbf{v},\mathbf{u})_{1/2,S} = (\mathbf{T}\mathbf{v},\mathbf{T}\mathbf{u})_{-1/2,S} \tag{2.3}$$

L e m m a 1. The operator $\mathbf{T}^*\mathbf{T}$ acting in $W_2^{1/2}(S)$ is self-adjoint and positive.

The proof follows from the fact that the domain of definition is $D(\mathbf{T}^*\mathbf{T}) = W_2^{1/2}(S)$ and the operator $\mathbf{T}$ is closed and positive.

Let us define a new scalar product

$$[\mathbf{u},\mathbf{v}]_{1/2,S} = (\mathbf{u},\mathbf{T}^*\mathbf{T}\mathbf{v})_{1/2,S} \tag{2.4}$$

with the norm $|\mathbf{u}|_{1/2,S} = \{[\mathbf{u},\mathbf{u}]_{1/2,S}\}^{1/2}$, for the vector functions $\mathbf{u},\mathbf{v}\in W_2^{-1/2}(S)$. Since the absolute value of the operator $\mathbf{T}$ is defined in the form $|\mathbf{T}| = \sqrt{\mathbf{T}^*\mathbf{T}}$, then the scalar product can be defined also as follows:

$$[\mathbf{u},\mathbf{v}]_{1/2,S} = (|\mathbf{T}|\mathbf{u},|\mathbf{T}|\mathbf{v})_{1/2,S} \tag{2.5}$$

The Hilbert space thus obtained which is isometric to the space $W_2^{1/2}(S)$, as follows from (2.5), will be denoted by $W_2^{*1/2}(S)$. There follows from the isometry of the spaces $W_2^{*1/2}(S)$ and $W_2^{1/2}(S)$ that the space conjugate to $W_2^{*1/2}(S)$ is $W_2^{-1/2}(S)$. There results from (2.3) and (2.4) that the spaces $W_2^{*1/2}(S)$ and $W_2^{-1/2}(S)$ are isometric

$$[\mathbf{u},\mathbf{v}]_{1/2,S} = (\mathbf{T}\mathbf{u},\mathbf{T}\mathbf{v})_{-1/2,S}, \quad \forall\mathbf{u},\mathbf{v}\in W_2^{*1/2}(S)$$

Therefore, the operator $\mathbf{T} : W_2^{*1/2}(S) \to W_2^{-1/2}(S)$ is isometric and unitary because of the reversibility (the core of the operator $\mathbf{T}$ in $W_2^{*1/2}(S)$ is $\ker \mathbf{T} = \{0\}$). Therefore, the space constructed $W_2^{*1/2}(S)$ is a subspace of the traces $\mathbf{u}|_S \in W_2^{1/2}(S)$ for which the boundary form $J_S(\mathbf{u}, \mathbf{u})$ is positive. It follows from the results of Sect. 1 that the boundary values $\varphi_0|_S$ of the solution φ_0 of the additional problem (1.2) in the sense of (1.3) belong to this subspace.

It follows from the theorem on the polar decomposition of bounded operators [5] that the polar decomposition $\mathbf{T} = \mathbf{U}\,|\mathbf{T}|$ holds for the operator $\mathbf{T}$ acting from $W_2^{1/2}(S)$ into $W_2^{-1/2}(S)$, where $|\mathbf{T}| = \sqrt{\mathbf{T}^*\mathbf{T}}$ is a positive operator acting in $W_2^{1/2}(S)$, and $\mathbf{U}$ is a partial isometry from $W_2^{1/2}(S)$ in $W_2^{-1/2}(S)$, which is defined uniquely by the condition $\ker \mathbf{U} = \ker \mathbf{T}$. Since the operator is reversible, then the operator $\mathbf{U}$ in the decomposition presented is unitary [5].

L e m m a 2. The operator is $\mathbf{U} = \mathbf{T}_0$ in the polar decomposition $\mathbf{T} = \mathbf{U}\,|\mathbf{T}|$, where $\mathbf{T}_0$ is the canonical isometry of $W_2^{1/2}(S)$ into $W_2^{-1/2}(S)$.

The proof results from the fact that the operators $\mathbf{U}$ and $\mathbf{T}_0$ isometrically map the domain of values of the operator $|T|$: $\operatorname{Ran}|\mathbf{T}| = W_2^{*1/2}(S)$ into $W_2^{-1/2}(S)$, and therefore, the following representations hold

$$[\mathbf{u}, \mathbf{v}]_{1/2,\,S} = (\mathbf{u}, U\mathbf{v})_{0,\,S}, \quad [\mathbf{u}, \mathbf{v}]_{1/2,\,S} = (\mathbf{u}, T_0\mathbf{v})_{0,\,S}, \quad \forall \mathbf{u}, \mathbf{v} \in W_2^{*1/2}(S)$$

Examining these equalities jointly, we obtain that $\mathbf{U} = \mathbf{T}_0$. The propositions a) and b) formulated in Sect. 1 are proved on the basis of the polar decomposition $\mathbf{T} = \mathbf{T}_0\,|\mathbf{T}|$.

T h e o r e m 1. The operator $\mathbf{T}$ is a partial isometry from the space $W_2^{1/2}(S)$ into the space $W_2^{-1/2}(S)$ which satisfies the condition $\ker \mathbf{T} = \ker \mathbf{T}_0$ with the intial space $(\ker \mathbf{T})^\perp = W_2^{*1/2}(S)$.

The proof follows from the fact that the equality $\mathbf{T}\mathbf{u} = \mathbf{T}_0\mathbf{u}$ holds in the vector functions $\mathbf{u} \in \operatorname{Ran}|\mathbf{T}| = W_2^{*1/2}(S)$, and, predefining the operator T_0 to be zero in the orthogonal complement $(\operatorname{Ran}|\mathbf{T}|)^\perp$, we obtain $\ker \mathbf{T} = \ker \mathbf{T}_0 = (\operatorname{Ran}|\mathbf{T}|)^\perp$ because of the equalities $(\operatorname{Ran}|\mathbf{T}|)^\perp = \ker|\mathbf{T}|$ (which follows from the self-adjointness of the operator $|\mathbf{T}| = |\mathbf{T}|^*$) and $\ker \mathbf{T} = \ker|\mathbf{T}|$.

C o r o l l a r y. The orthogonal expansion (1.7) of the space $W_2^{1/2}(S)$ holds, and the operator $\mathbf{T}^*\mathbf{T}$ is a projector on $W_2^{*1/2}(S)$.

Since the spaces $W_2^{*1/2}(S)$ and $W_2^{-1/2}(S)$ are isometric, then by the Riesz theorem the relationship (see (1.6))

$$[\mathbf{u}, \mathbf{v}]_{1/2,\,S} = (\mathbf{u}, \mathbf{T}\mathbf{v})_{0,\,S} = (\mathbf{T}\mathbf{u}, \mathbf{T}\mathbf{v})_{-1/2,\,S}, \quad \forall \mathbf{u}, \mathbf{v} \in W_2^{*1/2}(S) \tag{2.6}$$

holds. From this and (2.2) there follows

$$[\mathbf{u}, \mathbf{v}]_{1/2,\,S} = J_S(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in W_2^{*1/2}(S) \tag{2.7}$$

From (2.7) there follows that

1°. The bilinear form $J_S(\mathbf{u}, \mathbf{v})$, $\forall \mathbf{u}, \mathbf{v} \in W_2^{*1/2}(S)$ can be considered as a generalization of the fractional scalar product in the space $W_2^{1/2}(S)$ in boundary value problems of elasticity theory;

2°. If the vectors $\mathbf{u} \in W_2^{1/2}(S)$ and $\mathbf{t}^{(\nu)}(\mathbf{v}) \in W_2^{-1/2}(S)$ make the integral

$J_S(\mathbf{u}, \mathbf{v})$ vanish, then the vectors $\mathbf{u}|_S$ and $\mathbf{v}|_S$ are orthogonal in the metric of the space of traces $W_2^{*1/2}(S)$.

3. Orthogonal expansions on the domain boundary in elasticity theory problems. Let us formulate the condition for orthogonality of the traces $\mathbf{u}'|_S$ and $\mathbf{u}''|_S$ of the displacement vectors $\mathbf{u}'$ and $\mathbf{u}''$ associated with the tensors $\mathbf{R}'$ and $\mathbf{R}''$ and making the boundary integral $J_S(\mathbf{u}', \mathbf{u}'')$ vanish, in the metric $W_2^{*1/2}(S)$. Let us prove some auxiliary propositions.

Lemma 3. Let $P \subseteq W_2^{1/2}(S)$ be a closed subspace of traces of the vectors $\mathbf{u}'$. Then the linear set of functionals

$$P^{\perp} = \{\mathbf{t}^{(\nu)}(\mathbf{u}'') \in W_2^{-1/2}(S) \,|\, \langle \mathbf{u}', \mathbf{t}^{(\nu)}(\mathbf{u}'')\rangle = 0, \quad \forall \mathbf{u}' \in P\}$$

is a closed subspace of the space $W_2^{-1/2}(S)$.

Let $\{\mathbf{t}_n^{(\nu)}(\mathbf{u}'')\}$ be a sequence in $P^{\perp}$ for which $\| \mathbf{t}_n^{(\nu)}(\mathbf{u}'') - \mathbf{t}^{(\nu)}(\mathbf{u}'')\|_{-1/2, S} \underset{n\to\infty}{\to} 0$ and $\mathbf{t}^{(\nu)}(\mathbf{u}'') \in W_2^{-1/2}(S)$, then $|\langle \mathbf{u}', \mathbf{t}_n^{(\nu)}(\mathbf{u}'') - \mathbf{t}^{(\nu)}(\mathbf{u}'')\rangle| \leqslant \|\mathbf{u}'\|_{1/2, S} \| \mathbf{t}_n^{(\nu)}(\mathbf{u}'') - \mathbf{t}^{(\nu)}(\mathbf{u}'')\|_{-1/2, S} \underset{n\to\infty}{\to} 0$. Therefore $\mathbf{t}^{(\nu)}(\mathbf{u}'') \in P^{\perp}$.

We call $P^{\perp}$ the orthogonal complement to P in $W_2^{-1/2}(S)$.

Lemma 4. The linear set

$$P^{\oplus} = \{\mathbf{u}'' \in W_2^{*1/2}(S) \,|\, [\mathbf{u}', \mathbf{u}'']_{1/2, S} = 0, \quad \forall \mathbf{u}' \in P\}$$

is a closed subspace of the space $W_2^{*1/2}(S)$ and is a Hilbert orthogonal complement to P in $W_2^{*1/2}(S)$.

Lemma 4 is proved analogously to Lemma 3.

Lemma 5. The relation between the subspaces $P^{\oplus}$ and $P^{\perp}$ of the dual pair $W_2^{*1/2}(S) \times W_2^{-1/2}(S)$ is established according to the relationship $P^{\perp} = \mathbf{T}P^{\oplus}$, where $\mathbf{T}$ is the isometry $W_2^{*1/2}(S)$ into $W_2^{-1/2}(S)$.

Indeed, since by virtue of the equalities (2.6) and (2.7)

$$[\mathbf{u}', \mathbf{u}'']_{1/2, S} = (\mathbf{u}', \mathbf{T}\mathbf{u}'')_{0, S} = \langle \mathbf{u}', \mathbf{t}^{(\nu)}(\mathbf{u}'')\rangle, \quad \forall \mathbf{u}' \in W_2^{*1/2}(S)$$

then $\mathbf{u}'' \in P^{\oplus}$ if and only if $\mathbf{T}\mathbf{u}'' \in P^{\perp}$, i.e., when $\mathbf{t}^{(\nu)}(\mathbf{u}'') \in P^{\perp}$.

The subspace P will be treated as a subspace of the boundary conditions for the displacement vectors $\mathbf{u}'$ on the clamped part of the surface S; the subspace $P^{\oplus}$ is treated as a subspace of the boundary values of the vectors $\mathbf{u}''$ which satisfy the homogeneous elasticity theory equation $\mathbf{A}\mathbf{u}'' = 0$ and the boundary conditions on the free part of the surface in the domain G.

Theorem 2. If the vectors $\mathbf{u}'|_S \in P$, $\mathbf{t}^{(\nu)}(\mathbf{u}'') \in P^{\perp}$, i.e., are orthogonal in the sense $\langle \mathbf{u}', \mathbf{t}^{(\nu)}(\mathbf{u}'')\rangle = 0$, then the vectors $\mathbf{u}'|_S \in P$, $\mathbf{u}''|_S \in P^{\oplus}$, i.e., are orthogonal in the sense $[\mathbf{u}', \mathbf{u}'']_{1/2,S} = 0$.

Theorem 2 results from Lemmas 3–5.

Corollary. The orthogonality condition $[\mathbf{u}_0, \boldsymbol{\varphi}_0]_{1/2,S} = 0$ holds for the expansion $\mathbf{u}^* = \mathbf{u}_0 + \boldsymbol{\varphi}_0$ (see Sect. 1).

Theorem 3. An arbitrary displacement vector $\mathbf{u}^*$ satisfying the elasticity theory equation $\mathbf{Au}^* = \mathbf{K}$ and the boundary condition on the free part of the surface S such that $\mathbf{u}^*|_S \in W_2^{1/2}(S)$ will be represented in the form $\mathbf{u}^*|_S = \mathbf{u}'|_S + \mathbf{u}''|_S$, $\mathbf{u}'|_S \in P$, $u''|_S \in P^{\oplus}$ on the boundary S.

Let the stress vector $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$ associated with $\mathbf{u}^*(x)$ vanish on the free part of the surface S. Let $\mathbf{u}'$ be the vector of the elastic displacements, then $\mathbf{u}'|_S \in P$ and $\mathbf{t}^{(\nu)}(\mathbf{u}') \in P^{\perp}$. Let us set $\mathbf{u}'' = \mathbf{u}^* - \mathbf{u}'$. Then the vector $\mathbf{u}''$ satisfies the homogeneous equation of elasticity theory. We show that $\mathbf{u}''|_S \in P^{\oplus}$. Indeed, since the relationship $\langle \mathbf{u}', \mathbf{t}^{(\nu)}(\mathbf{u}^*)\rangle = J_S(\mathbf{u}', \mathbf{u}^*) = 0$, holds for the boundary conditions of elasticity theorem problems, then by virtue of Lemma 3 the vector $\mathbf{t}^{(\nu)}(\mathbf{u}^*) \in P^{\perp}$. Then the vector $\mathbf{t}^{(\nu)}(\mathbf{u}'') = \mathbf{t}^{(\nu)}(\mathbf{u}^*) - \mathbf{t}^{(\nu)}(\mathbf{u}') \in P^{\perp}$. Hence $\mathbf{u}''|_S \in P^{\oplus}$ follows by virtue of Lemma 5.

C o r o l l a r y. The orthogonal expansion

$$W_2^{*1/2}(S) = P \oplus P^{\oplus}.$$

follows from Theorems 2 and 3.

Since $\mathbf{T}$ is a bounded operator from $W_2^{1/2}(S)$ into $W_2^{-1/2}(S)$, then the expression $(\mathbf{u}, \mathbf{Tv})_{0,S}$, $\forall \mathbf{u}, \mathbf{v} \in W_2^{1/2}(S)$ is a bilinear continuous functional in $W_2^{1/2}(S)$. Then the bounded linear operator $\mathbf{T}^{\otimes}$ which also maps $W_2^{1/2}(S)$ into $W_2^{-1/2}(S)$ and for which $(\mathbf{u}, \mathbf{Tv})_{0,S} = (\mathbf{T}^{\otimes}\mathbf{u}, \mathbf{v})_{0,S}$, $\forall \mathbf{v} \in W_2^{1/2}(S)$, is determined uniquely. We call the operator $\mathbf{T}^{\otimes}$ the generalized conjugate to the operator $\mathbf{T}$. If $\mathbf{u}, \mathbf{v} \in W_2^{*1/2}(S)$, then $[\mathbf{u}, \mathbf{v}]_{1/2,S} = (\mathbf{u}, \mathbf{Tv})_{0,S} = (\overset{\otimes}{\mathbf{T}}\mathbf{u}, \mathbf{v})_{0,S}$ follows from (2.6), i.e., the operator $\mathbf{T}$ defined on $W_2^{*1/2}(S)$ is generally self-adjoint $\mathbf{T} = \mathbf{T}^{\otimes}$ (the operator $\mathbf{T}^{\otimes}$ has the meaning of a self-adjoint Friedrichs expansion of the operator $\mathbf{T}$).

L e m m a 6. $(\mathrm{Ran}\,\mathbf{T})^{\perp} = \ker \mathbf{T}$.

The proof of the lemma follows from the exposition above (see [5], for instance).

Let us now specify the subspaces P and $P^{\oplus}$ for the first and second boundary value problems of elasticity theory [1].

The first boundary value problem is: $\mathbf{u}'|_S = \mathbf{u}_0|_S = 0$, then $P = \{0\}$ and, therefore, $\langle \mathbf{u}', \mathbf{t}^{(\nu)}(\mathbf{u}'')\rangle = (\mathbf{u}', \mathbf{Tu}'')_{0,S} = 0$, $\forall \mathbf{t}^{(\nu)}(\mathbf{u}'') \in W_2^{-1/2}$, i.e., $P^{\perp} = W_2^{-1/2}$. Then $\mathbf{u}' \in (\mathrm{Ran}\,\mathbf{T})^{\perp} = \ker \mathbf{T} = P = \{0\}$ follows from Lemma 6, and $P^{\oplus} = W_2^{*1/2}(S) = (\ker \mathbf{T})^{\perp}$ follows from $P^{\oplus} = \mathbf{T}^{-1}P^{\perp}$. Therefore the vectors $\mathbf{u}'' = \varphi_0$, $\mathbf{t}^{(\nu)}(\mathbf{u}'') = \mathbf{t}^{(\nu)}(\varphi_0)$ are not subject to any boundary conditions on S since the whole boundary is clamped.

The second boundary value problem is: $\mathbf{t}^{(\nu)}(\mathbf{u}'')|_S = \mathbf{t}^{(\nu)}(\varphi_0)|_S = 0$; then $P^{\perp} = \{0\}$ and $\langle \mathbf{u}', \mathbf{t}^{(\nu)}(\mathbf{u}'')\rangle = (\mathbf{u}', \mathbf{Tu}'')_{0,S} = 0$, $\forall \mathbf{u}' \in W_2^{1/2}$, i.e., $P = W_2^{1/2}(S)$ (indeed, the vector $\mathbf{u}' = \mathbf{u}_0$ is not subject to any boundary conditions in the case of the second boundary value problem). From the relations $P^{\oplus} = \mathbf{T}^{-1}P^{\perp}$ and Lemma 6 there follows $\mathbf{u}'' \in P^{\oplus} = \{0\} = \ker \mathbf{T}$, therefore, the vector $\mathbf{u}'' = \varphi_0 = 0$ on S.

In fact, if $\mathbf{t}^{(\nu)}(\mathbf{u}^*)|_S = 0$, then the fact that the volume integral for all the vectors $\mathbf{u} \in W_2^1(G)$ is zero (such that $\varepsilon_{lm}(\mathbf{u}) \neq 0$) follows from the integral identity (1.3). Then $\varphi_0 = \text{const}$, i.e., φ_0 is the vector of a small rigid body displacement without deformation, which is impossible since the constant satisfying condition (1.5) is identically zero.

In conclusion, let us indicate the relationship between the exposition and the Castigliano principle, which is that the norms of the stress vectors $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$, $\mathbf{t}^{(\nu)}(\mathbf{u}_0)$, $\mathbf{t}^{(\nu)}(\varphi_0)$ acting on the surface S are interrelated by an inequality analogous to (1.1). In fact, the orthogonality of the elements $\mathbf{T}\mathbf{u}_0$ and $\mathbf{T}\varphi_0$ in the metric $W_2^{-1/2}(S)$ follows from the orthogonality of the traces $\mathbf{u}_0|_S$ and $\varphi_0|_S$ in the metric $W_2^{*1/2}(S)$ and the equality (2.6). Then the inequality

$$\| \mathbf{t}^{(\nu)}(\mathbf{u}^*) \|^2_{-1/2, S} = \| \mathbf{t}^{(\nu)}(\mathbf{u}_0) \|^2_{-1/2, S} + \| \mathbf{t}^{(\nu)}(\mathbf{u}^*) - \mathbf{t}^{(\nu)}(\mathbf{u}_0) \|^2_{-1/2, S} \geqslant \| \mathbf{t}^{(\nu)}(\mathbf{u}_0) \|^2_{-1/2, S}$$

holds because of the equality $\|\mathbf{T}\mathbf{u}\|_{-1/2,S} = \| \mathbf{t}^{(\nu)}(\mathbf{u}) \|_{-1/2,S}$.

4. Construction of the solution of the fundamental boundary value problems of elasticity theory. The first problem is: $\mathbf{A}\mathbf{u}_0 = \mathbf{K}$ in G, $\mathbf{u}_0|_S = 0$.

1° From the results of Sect. 3, $P = \{0\}$, $P^{\oplus} = W_2^{*1/2}(S)$ for the first problem. Then using the scheme of the method elucidated in Sect. 1, we obtain the solution $\mathbf{u}_0$ by projecting the vector $\mathbf{u}^*$ on the subspace $P^{\oplus} = W_2^{*1/2}(S)$ and subtracting the projection from the vector $\mathbf{u}^* : \mathbf{u}_0 = \mathbf{u}^* - \mathbf{\Pi}\mathbf{u}^* = \mathbf{u}^* - \varphi_0$, where $\mathbf{\Pi} = \mathbf{T}^*\mathbf{T}$ is the orthoprojector on $W_2^{*1/2}(S)$ (see the corollary to Theorem 1). Therefore, $\mathbf{u}_0|_S \in P = \{0\}$ on the boundary.

2°. As follows from Sect. 1, the vector φ_0 is a solution of the additional problem (1.2) in the sense of (1.3), which hence satisfies condition (1.5). Let us construct the projection $\varphi_0 = \mathbf{\Pi}\mathbf{u}^*$. Let $\{\psi_i\}_{i=1}^{i=\infty}$ be a system of linearly independent sufficiently smooth vector functions such that the ψ_i satisfy the equation $\mathbf{A}\psi_i = 0$ in the domain G and the condition

$$\int_G \psi_i dG = 0$$

Let us orthonormalize the system $\{\psi_i\}$ with respect to the energy of the second boundary value problem:

$$\frac{\psi_i}{|\psi_i|_{H_2}} = \psi_i^\circ, \quad [\psi_i^\circ, \psi_k^\circ]_{H_2} = 2\int_G W(\psi_i^\circ, \psi_k^\circ)\, dG = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \tag{4.1}$$

In fact [1], the expression

$$\left\{2\int_G W(\mathbf{u})dG\right\}^{1/2}$$

for $\mathbf{u} \in W_2^1(G)$ satisfying (1.5), is the energy norm of the second boundary value problem of elasticity theory. Let us subject the system $\{\psi_i^\circ\}$ to the condition of completeness with respect to the energy of the second problem [1]. Then, since ψ_i° satisfies $\mathbf{A}\psi_i^\circ = 0$ by virtue of the equalities (2.1), (2.7) and (4.1), the system $\{\psi_i^\circ\}$ is orthonormalized and complete in the metric of the space $W_2^{*1/2}(S)$. Let us construct the projection

$$\varphi_0 = \mathbf{\Pi}\mathbf{u}^* = \sum_{i=1}^{\infty} [\mathbf{u}^*, \psi_i^\circ]_{1/2, S}\psi_i^\circ = \sum_{i=1}^{\infty} J_S(\mathbf{u}^*, \psi_i^\circ)\, \psi_i^\circ$$

so that $\varphi_0|_S \in P^{\oplus}$, then $u_0|_S = (\mathbf{u}^* - \varphi_0)|_S \in P = \{0\}$.

3°. The vector $\varphi_{0i} = J_S(\mathbf{u}^*, \psi_i^\circ)\psi_i^\circ$, $i = 1, 2, \ldots$ satisfies the integral identity (1.3). Indeed, we obtain from (1.3)

$$J_S(\mathbf{u}^*, \psi_i^\circ) \cdot 2\int_G W(\psi_i^\circ, \mathbf{u})\, dG - J_S(\mathbf{u}^*, \mathbf{u}) = 0, \quad \forall \mathbf{u} \in W_2^1(G)$$

We now set $\mathbf{u} = \psi_i^\circ$, $i = 1, 2, \ldots$, and taking account of (4.1) obtain the identity.

4°. Let us show that the vector $\mathbf{u}_0 = \mathbf{u}^* - \varphi_0$ is a solution of the first problem of elasticity theory. To do this it is sufficient to see that $\mathbf{u}_0$ is the energy solution of this problem. Satisfaction of the boundary condition $\mathbf{u}_0|_S = 0$ follows from $\mathbf{u}_0|_S \in P = \{0\}$. The vector $\mathbf{u}^*$ which is a solution of the equation $A\mathbf{u}^* = \mathbf{K}$ satisfies the integral identity [2]:

$$2\int_G W(\mathbf{u}^*, \mathbf{u})\, dG - J_S(\mathbf{u}^*, \mathbf{u}) = \int_G \mathbf{K}\mathbf{u}\, dG, \quad \forall \mathbf{u} \in W_2^1(G).$$

The vector φ_0 satisfies (1.3), therefore, the vector $\mathbf{u}_0$ satisfies the integral identity

$$2\int_G W(\mathbf{u}_0, \mathbf{u})\, dG = \int_G \mathbf{K}\mathbf{u}\, dG, \quad \forall \mathbf{u} \in W_2^1(G)$$

which is a generalization of the Euler–Lagrange equation for the energy functional of the first problem [1] for $\mathbf{u} \in H_1 \subset W_2^1(G)$ (H_1 is the energy space of the first problem of elasticity theory [1]).

The second problem is: $A\mathbf{u}_0 = \mathbf{K}$ in G, $\mathbf{t}^{(\nu)}(\mathbf{u}_0)|_S = 0$. For the second problem $P = W_2^{1/2}(S)$, $P^\oplus = \{0\}$. From $P^\oplus = \{0\}$ there follows that $\varphi_0|_S = 0$ and the vector φ_0 is identically zero as a solution of the problem $A\varphi_0 = 0$, $\mathbf{t}^{(\nu)}(\varphi_0)|_S = 0$ ($\mathbf{t}^{(\nu)}(\varphi_0)|_S = \mathbf{t}^{(\nu)}(\mathbf{u}^*)|_S = 0$ is the condition on the free surface S). Then $\mathbf{u}_0 = \mathbf{u}^* - \varphi_0 = \mathbf{u}^*$, and therefore, $A\mathbf{u}_0 = \mathbf{K}$ in G; on the boundary $\mathbf{u}_0|_S \in P = W_2^{1/2}(S)$. It hence follows that the vector $\mathbf{u}^*$ can itself be a solution of the second problem since

$$A\mathbf{u}^* = \mathbf{K}, \ \mathbf{t}^{(\nu)}(\mathbf{u})^*|_S = 0.$$

The third problem is: $A\mathbf{u}_0 = \mathbf{K}$ in G, $\mathbf{u}_0|_{S_1} = 0$, $\mathbf{t}^{(\nu)}(\mathbf{u}_0)|_{S_2} = 0$. The vector

$$\mathbf{u}_0 = \mathbf{u}^* - \sum_{i=1}^{\infty} [\mathbf{u}^*, \psi_i^\circ]_{1/2, S_1}\psi_i^\circ = \mathbf{u}^* - \sum_{i=1}^{\infty} J_{S_1}(\mathbf{u}^*, \psi_i^\circ)\psi_i^\circ$$

is the solution of this problem (here the projector $\Pi_1 = \mathbf{T}_1^*\mathbf{T}_1$ is determined just by the integral over S_1). Therefore $\mathbf{u}_0|_{S_1} = (\mathbf{u}^* - \varphi_0)|_{S_1} \in P_{S_1} = \{0\}$, hence the vector $\mathbf{u}^*$ and the coordinate functions ψ_i must be subjected additionally to the condition $\mathbf{t}^{(\nu)}(\psi_i)|_{S_1} = 0$.

As follows from [8], the vector $\mathbf{u}^*$ can be taken as

$$\mathbf{u}^*(x) = \int_G \mathbf{V}\mathbf{K}\, dG$$

where $\mathbf{V}$ is the Somigliana tensor [8].

REFERENCES

1. Mikhlin, S. G., Variational Methods in Mathematical Physics. (English translation), Pergamon Press, Book No. 10146, 1964.

2. Tereshchenko, V. Ia., Generalization of the Trefftz method for three-dimensional elasticity problems, (English translation), Pergamon Press, J. USSR Comput. Math. math. Phys. Vol. 16, No. 4, 1976.

3. Lions, J.-L. and Magenes, E. Problèmes aux dimites non Homogènes et Applications, Paris, Dunod, 1968-1970.

4. Slobodetskii, L. N. Generalized S. L. Sobolev spaces and their application to boundary value problems for partial differential equations, Uchen. Zapiski Leningrad State Pedagogic Inst., Vol. 197, 1958.

5. Dunford, N. and Schwartz, J., Linear Operators, Pt. 2 Spectral Theory. "Mir," Moscow, 1966 (see also pt. 1, New York Interscience Publ. Incorp. 1958).

6. Vishik, M. I., Method of orthogonal and direct expansions in the theory of elliptic differential equations, Matem. Sb., Vol. 25, No. 2, 1949.

7. Akhiezer, N. I. and Glazman, I. M., Theory of Linear Operators in Hilbert Space. "Nauka", Moscow, 1966.

8. Mikhlin, S. G., Problem of the Minimum of a Quadratic Functional. (English translation), San Francisco, Holden - Day, 1965.

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